Computational Hydraulics
Higher Order Schemes: QUICKEST

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The QUICKEST (Quadratic Upstream Interpolation for Convective Kinematics with Estimated Streaming Terms) Scheme is a third order accurate explicit finite difference scheme developed by Leonard (1979) to solve the advection-diffusion equation. The scheme is upwinded hence is only valid in the form presented here if $u > 0$.

QUICKEST uses the first four terms in the Taylor expansion of *c* in the forward time direction:

$$c_{i,j+1} = c_{i,j} + \Delta t \frac{\partial c}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 c}{\partial t^2} + \frac{\Delta t^3}{6}\frac{\partial^3 c}{\partial t^3} + \mathrm{O}(\Delta t^4) \qquad (1)$$

and uses five calculation points on the finite difference grid as illustrated in Figure 1.

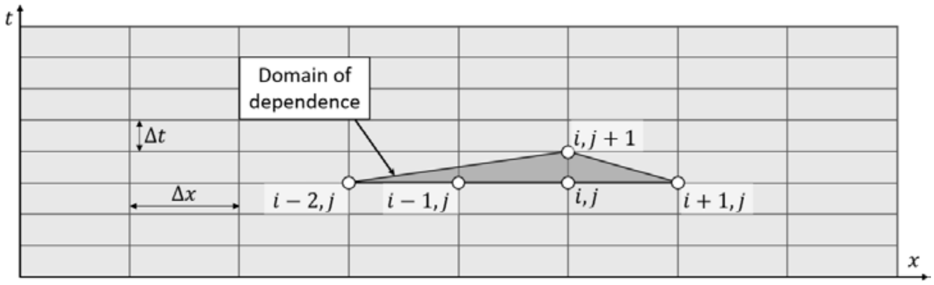

Figure 1. Calculation points used in the QUICKEST scheme

QUICKEST Scheme for Advection

Consider first the case of advection only, given by the equation:

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = 0$$

where $c = c(x,t)$ is the concentration of a dissolved substance (mg/m^3) and u is the flow celerity (m/s) and $u > 0$.

First we note that the advection equation yields the following results:

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x} \Rightarrow \frac{\partial^2 c}{\partial t^2} = u^2\frac{\partial^2 c}{\partial x^2} \quad \text{and} \quad \frac{\partial^3 c}{\partial t^3} = -u^3\frac{\partial^3 c}{\partial x^3}$$

Substituting into equation (1) we have:

$$c_{i,j+1} = c_{i,j} - u\Delta t\frac{\partial c}{\partial x} + \frac{(u\Delta t)^2}{2}\frac{\partial^2 c}{\partial x^2} - \frac{(u\Delta t)^3}{6}\frac{\partial^3 c}{\partial x^3} + \mathrm{O}(\Delta t^4) \qquad (2)$$

The scheme uses the following approximation to the derivatives:

(a) a third order upwinded finite difference approximation to the first derivative ($\mathrm{O}(\Delta x^3)$):

$$\frac{\partial c}{\partial x} \approx \frac{c_{i-2,j} - 6c_{i-1,j} + 3c_{i,j} + 2c_{i+1,j}}{6\Delta x} \qquad (3)$$

(b) a second order central difference approximation to the second derivative ($\mathrm{O}(\Delta x^2)$):

$$\frac{\partial^2 c}{\partial x^2} \approx \frac{c_{i-1,j} - 2c_{i,j} + c_{i+1,j}}{\Delta x^2} \qquad (4)$$

(c) a first order upwinded finite difference approximation to the third derivative ($O(\Delta x)$):

$$\frac{\partial^3 c}{\partial x^3} \approx \frac{-c_{i-2,j} + 3c_{i-1,j} - 3c_{i,j} + c_{i+1,j}}{\Delta x^3} \tag{5}$$

When these are substituted into the expansion for $c_{i,j+1}$ the local truncation error is of order $\Delta t \Delta x^3 + \Delta t^2 \Delta x^2 + \Delta t^3 \Delta x + \Delta t^4$.

Substituting approximations (3), (4) and (5) into Equation (2) yields

$$c_{i,j+1} = c_{i,j} - \tfrac{1}{6}\rho\left(c_{i-2,j} - 6c_{i-1,j} + 3c_{i,j} + 2c_{i+1,j}\right) + \tfrac{1}{2}\rho^2\left(c_{i-1,j} - 2c_{i,j} + c_{i+1,j}\right)$$
$$-\tfrac{1}{6}\rho^3\left(-c_{i-2,j} + 3c_{i-1,j} - 3c_{i,j} + c_{i+1,j}\right)$$

where $\rho = u\frac{\Delta t}{\Delta x}$ is the Courant number

Rearranging this gives the QUICKEST scheme for the advection equation (Leonard, 1979):

$$c_{i,j+1} = \tfrac{1}{6}\rho(\rho^2 - 1)c_{i-2,j} - \tfrac{1}{2}\rho(\rho^2 - \rho - 2)c_{i-1,j} + \left(1 + \tfrac{1}{2}\rho(\rho^2 - 2\rho - 1)\right)c_{i,j}$$
$$- \tfrac{1}{6}\rho(\rho^2 - 3\rho + 2)c_{i+1,j}$$

QUICKEST Scheme for Advection and Diffusion

The advection-diffusion equation is given by

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = D\frac{\partial^2 c}{\partial x^2}$$

where $c = c(x,t)$ is the concentration of a dissolved substance (mg/m^3) and u is the flow celerity (m/s) ($u > 0$) as before, and D is the diffusion coefficient (m^2/s) ($D > 0$).

In this case we have

$$\frac{\partial c}{\partial t} = -u\frac{\partial c}{\partial x} + D\frac{\partial^2 c}{\partial x^2}$$

which yields

$$\frac{\partial^2 c}{\partial t^2} = u^2\frac{\partial^2 c}{\partial x^2} - 2uD\frac{\partial^3 c}{\partial x^3} + D^2\frac{\partial^4 c}{\partial x^4} \quad \text{and} \quad \frac{\partial^3 c}{\partial t^3} = -u^3\frac{\partial^3 c}{\partial x^3} + 3u^2 D\frac{\partial^4 c}{\partial x^4} - 3uD^2\frac{\partial^5 c}{\partial x^5} + D^3\frac{\partial^6 c}{\partial x^6}$$

Substituting approximations (3), (4) and (5) into Equation (2) and ignoring fourth and higher order derivatives gives the QUICKEST scheme for transport and diffusion (Leonard, 1979):

$$c_{i,j+1} = \rho\left(r + \tfrac{1}{6}(\rho^2 - 1)\right)c_{i-2,j} + \left(r(1 - 3\rho) - \tfrac{1}{2}\rho(\rho^2 - \rho - 2)\right)c_{i-1,j}$$
$$+ \left(1 - r(2 - 3\rho) + \tfrac{1}{2}\rho(\rho^2 - 2\rho - 1)\right)c_{i,j} + \left(r(1 - \rho) - \tfrac{1}{6}\rho(\rho^2 - 3\rho + 2)\right)c_{i+1,j}$$

Reference

Leonard, B.P. (1979), "A stable and accurate convective modelling procedure based on quadratic upstream interpolation", *Computer Methods in Applied Mechanics and Engineering*, **19**(1): 59–98, doi:10.1016/0045-7825(79)90034-3